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Brownian motion of electrons in time-dependent magnetic fields

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Abstract. The behaviour of a weakly ionized plasma in slowly varying time-dependent magnetic fields is studied through an extension of Williamson's stochastic theory. In particular, attention is focused on the properties of electron diffusion in the plane perpendicular to the direction of the magnetic field, when the field strength is large. It is shown that, in the strong field limit, the classical $1/B^2$ dependence of the perpendicular diffusion coefficient is obtained for two models in which the field $B(t)$ is monotonic in t and for two models in which $B(t)$ possesses at least one turning point.

1. Introduction

Through the fundamental work of Kurşunoğlu (1963), Williamson (1968) and others (see also references contained in Kurşunoğlu (1963)), the adaptation of the classical theory of brownian motion to plasma physics has been successfully achieved (Williamson (1968) contains a review of other theories of brownian motion applicable to plasmas). There are several equivalent descriptions of the stochastic theory of brownian motion each of which one might attempt to generalize to describe the motion of particles in a plasma. The most obvious way to effect such a generalization is to introduce the Lorentz force into the Langevin equation and to solve the resulting stochastic differential equation under certain natural assumptions concerning the statistical nature of the fluctuating electric field; these assumptions provide 'boundary conditions' which yield a unique solution for the distribution function. This technique has been employed with great clarity by Kurşunoğlu (1963) in his study of plasma diffusion in a constant external magnetic field.

The method of Williamson (1968) involves the construction of a simple algebraic equation for the spectrum of the probability density of the net displacement. This method is familiar from the well known theory of stochastic processes as elucidated by Chandrasekhar (1943). Section 2 contains a brief description of the basic elements of this approach.

In this work we investigate the diffusion of a weakly ionized plasma under the influence of an external time-dependent magnetic field. If one sets up the generalized Langevin equation involving time variations of the magnetic field it is soon realized that the problem of solving this equation is by no means a simple task. On the other hand, we show in § 2 that the coefficient of diffusion can be calculated in a very simple way once we have solved the free particle equations of motion. In a sense therefore, the problem has been reduced to orbit theory. Indeed, we show that once the function

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$y(t) = (1/\dot{x}(0))(x(t)-x(0))$ is known the perpendicular diffusion coefficient is given by $D_\perp = (KT\gamma^2/2m)\int_0^\infty e^{-\gamma t}|y(t)|^2\, dt$, γ being the electron–neutral atom collision rate.

We are mainly interested here in discerning the effect of the time dependence of the impressed magnetic field on $D_\perp$. Specifically, writing $B(t) = BF(t)$, $B(0) = B$, we investigate the dependence of the perpendicular diffusion coefficient on B for $B \gg 1$, under the assumption that $F(t)$ is a very slowly varying function of t. In general, it is not possible to obtain precise quantitative results. However, in §§ 3 and 4, we consider models with specific functions $F(t)$ for which the equations of motion can be integrated. Some of these models are similar to the work of Seymour *et al* (1965) and Seymour (1966) in their description of charged particle motion in a solenoid.

Section 3 contains two models for which $F(t)$ is monotonic. In § 4 we consider two models in which the time-dependent fields possess turning points, that is, $\dot{F}(t) = 0$ for some $t > 0$. In all cases diffusion across the magnetic fields is characterized by the classical $1/B^2$ dependence. Section 5 consists of some concluding remarks.

2. Brownian motion in a time-dependent magnetic field

Williamson (1968) has recently shown that the electron component of a weakly ionized plasma can be described in a manner which is formally very similar to classical brownian motion. This is despite the fact that each electron–neutral atom collision is quite significant, contrary to the situation for brownian particles. It is convenient to record here some of the basic results of Williamson's theory. If $S(k, \omega)$ denotes the spectrum of the distribution function $W(x, t)$ of displacement Δx in an arbitrary direction, it is shown (Williamson 1968) that S obeys the algebraic equation

$$S = S_0(1-\gamma S_0)^{-1} \tag{2.1}$$

where γ is the electron–neutral atom collision rate and S_0 is the spectrum of the distribution function $W_0(x, t)$, defined as the probability of an electron travelling a distance Δx in time t without collision, that is,

$$W_0(x, t)\, dx = e^{-\gamma t} f(v_x, v)\, dv_x.$$

Here $f(v_x, v)$ is (ignoring inelastic processes) the rectangular velocity distribution

$$f(v_x, v) = \begin{cases} \dfrac{1}{2v}, & |v_x| \leqslant v \\ 0, & |v_x| > v \end{cases} \tag{2.2}$$

and we have denoted by v and v_x the length and x component of the velocity vector respectively. We have made the usual assumption that the scattering is isotropic.

The mean square displacement $\langle(\Delta x)^2\rangle$ attains a very simple form in terms of the spectrum. Indeed,

$$\langle(\Delta x)^2\rangle = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial^2 \bar{S}}{\partial k^2}\bigg|_{k=0} e^{i\omega t}\, d\omega. \tag{2.3}$$

$\bar{S}(k, \omega)$ is the spectrum $S(k, \omega)$ averaged over an assumed maxwellian velocity distribution $f_0(v)$:

$$\bar{S}(k, \omega) = \int_0^\infty S(k, \omega) f_0(v)\, dv. \tag{2.4}$$

The expression for $\partial^2 S/\partial k^2$ in terms of the derivatives of S_0 will be useful in the following. It is

$$\frac{\partial^2 S}{\partial k^2} = 2\gamma(1-\gamma S_0)^{-3}\left(\frac{\partial S_0}{\partial k}\right)^2 + (1-\gamma S_0)^{-2}\frac{\partial^2 S_0}{\partial k^2}. \tag{2.5}$$

We define the coefficient of diffusion D_x in the x direction by

$$D_x = \lim_{t\to\infty} \frac{1}{2t}\langle(\Delta x)^2\rangle. \tag{2.6}$$

Consider now the diffusion of electrons in a purely time-dependent magnetic field. For conciseness we take $\boldsymbol{B}(t) = (0, 0, B(t))$, and we shall assume that $B(t)$ is slowly varying, that is, $B(t)$ is quasi-static.

The time dependence of the field obviously necessitates an extension of the arguments used in deriving the equations above. Such an extension is formally necessary because the particle motion will now depend not only on the time interval required to travel a net distance, but will also be a function of the initial time τ. Equation (2.1) is then replaced by

$$\tilde{S}(k, \omega, \Omega) = \tilde{S}_0(k, \omega, \Omega) + \frac{\gamma}{2\pi}\int \mathrm{d}\Omega' \tilde{S}_0(k, \omega-\Omega', \Omega-\Omega')\tilde{S}(k, \omega, \Omega') \tag{2.1$'$}$$

where Ω is the Fourier variable conjugate to τ. However, the assumption that $B(t)$ is a very slowly varying function, implies that as a distribution in Ω,

$$\tilde{S}(k, \omega, \Omega) \simeq 2\pi\,\delta(\Omega)S(k, \omega)$$

where

$$S(k, \omega) = \frac{1}{2\pi}\int \mathrm{d}\Omega\tilde{S}(k, \omega, \Omega).$$

In this way, equation (2.1′) can be integrated to yield the original equation (2.1). Thus, in the adiabatic approximation, the τ dependence of the system can be integrated away.

Criteria for the adiabatic approximation are given by Seymour, who shows that it is sufficient to impose the conditions

$$\left|\frac{\dot{B}(t)}{B^2(t)}\right| \ll 1, \qquad \left|\frac{\ddot{B}(t)}{B^3(t)}\right| \ll 1, \qquad \text{for all } t > 0.$$

We shall consider only slowly varying fields which obey these restrictions.

In the calculation of the perpendicular diffusion coefficient, it suffices to consider a given direction in the plane perpendicular to the magnetic field. In order to calculate D_2, say, from equation (2.6), we need only obtain the displacement Δx_2 in time t of an electron undergoing free motion in the field $\boldsymbol{B}(t)$. This displacement is obtained, of course, by solving the equations of motion for such a system.

We write $B(t) = BF(t)$, $F(0) = 1$. Under the usual assumptions (Chandrasekhar 1960), the equations of motion in the 1–2 plane can be written

$$\ddot{x} = \mathrm{i}\omega(t)\dot{x} + \frac{\mathrm{i}}{2}\dot{\omega}(t)x \tag{2.7}$$

where $x(t) = x_1(t) + ix_2(t)$ and $\omega(t) = -eB(t)/m = \omega_c F(t)$. The initial conditions are given by

$$x(0) = \frac{v_\perp}{\omega_c} e^{i\psi}, \qquad \dot{x}(0) = iv_\perp e^{i\psi}. \tag{2.8}$$

Here, $v_\perp^2 = (v^2 - v_3^2)$ and ψ denotes the initial azimuthal angle in the 1–2 plane.

The 'normalized' displacement $y(t) = (1/\dot{x}(0))(x(t) - x(0))$ satisfies

$$\ddot{y}(t) = i\omega(t)\dot{y}(t) + \tfrac{1}{2}i\dot{\omega}(t)(y(t) - i\omega_c^{-1});$$
$$y(0) = 0, \qquad \dot{y}(0) = 1. \tag{2.8'}$$

Clearly $y(t)$ is independent of $v_\perp$ and ψ. The displacement along the 2 axis is thus given by

$$\Delta x_2 = \mathrm{Im}(x(t) - x(0)) = \mathrm{Im}(\dot{x}(0)y(t)) = v_\perp(\cos\psi\ \mathrm{Re}\ y(t) - \sin\psi\ \mathrm{Im}\ y(t)).$$

Clearly then

$$\exp(ik\,\Delta x_2) = \sum_{n,m=-\infty}^{\infty} i^n \mathrm{J}_{-m}(kv_\perp\ \mathrm{Im}\ y(t))\mathrm{J}_n(kv_\perp\ \mathrm{Re}\ y(t)) \exp\{i(m-n)\psi\}$$

where we have used the well known expansion of $\exp(ikz)$ in terms of Bessel functions (see eg Gradshteyn and Ryzhik 1965).

Now the spectrum S_0 can be written

$$S_0 = \frac{1}{2\pi}\int_0^\infty dt \int_{-\infty}^{\infty} dv_3 f(v, v_3) \int_0^{2\pi} d\psi \exp(ik\,\Delta x_2) \exp\{-(\gamma + i\omega)t\},$$

which by the above decomposition becomes

$$S_0 = \int_0^\infty dt \int_{-\infty}^{\infty} dv_3 f(v, v_3) \sum_{m=-\infty}^{\infty} i^m \mathrm{J}_{-m}(kv_\perp\ \mathrm{Im}\ y(t))\mathrm{J}_m(kv_\perp\ \mathrm{Re}\ y(t)) \exp\{-(\gamma + i\omega)t\}$$
$$= \int_0^\infty dt \int_{-\infty}^{\infty} dv_3 f(v, v_3)\mathrm{J}_0(kv_\perp|y(t)|) \exp\{-(\gamma + i\omega)t\}. \tag{2.9}$$

Equation (2.9) has been derived by making use of the addition theorem for Bessel functions (Gradshteyn and Ryzhik 1965).

It follows from equation (2.9) and the behaviour of J_0 and its derivatives at the origin that

$$S_0|_{k=0} = (\gamma + i\omega)^{-1}$$
$$\left.\frac{\partial S_0}{\partial k}\right|_{k=0} = 0 \tag{2.10}$$
$$\left.\frac{\partial^2 S_0}{\partial k^2}\right|_{k=0} = -\tfrac{1}{3}v^2 \int_0^\infty dt |y(t)|^2 \exp\{-(\gamma + i\omega)t\}.$$

From equations (2.5) and (2.10) we obtain

$$\left.\frac{\partial^2 \bar{S}}{\partial k^2}\right|_{k=0} = \frac{KT}{m}\frac{(\gamma + i\omega)^2}{\omega^2}\int_0^\infty dt|y(t)|^2 \exp\{-(\gamma + i\omega)t\}.$$

Here we have assumed (purely for simplicity) that γ is independent of v. This assumption allows the integration over v to be performed: $\int_0^\infty dv v^2 f_0(v) = 3KT/m$. We remark that this assumption is by no means drastic since it is well known that the coefficient of diffusion is fairly insensitive to any velocity dependence of the collision rate.

Using the above result in equation (2.3), we have

$$\begin{aligned}
&\langle(\Delta x_2)^2\rangle \\
&= \frac{-KT}{2m\pi}\int_0^\infty \mathrm{d}t'|y(t')|^2\exp(-\gamma t')\int_{-\infty}^\infty \mathrm{d}\omega\{\gamma^2(\omega-\mathrm{i}0)^{-2}+2\mathrm{i}\gamma(\omega-\mathrm{i}0)^{-1}-1\}\exp\{\mathrm{i}\omega(t-t')\} \\
&= \frac{KT}{m}\int_0^\infty \mathrm{d}t'|y(t')|^2\exp(-\gamma t')\{\gamma^2(t-t')\theta(t-t')+2\gamma\theta(t-t')+\delta(t-t')\} \\
&= \frac{KT}{m}\{|y(t)|^2\exp(-\gamma t)+2\gamma\int_0^t \mathrm{d}t'\exp(-\gamma t')(1-\tfrac{1}{2}\gamma t')|y(t')|^2+\gamma^2 t\int_0^t \mathrm{d}t'\exp(-\gamma t')|y(t')|^2\}.
\end{aligned} \tag{2.11}$$

In deriving (2.11) we have used the fact that $\int_0^\infty F(t)\exp(-\mathrm{i}\omega t)\,\mathrm{d}\omega$ is an analytic function in the lower half ω plane, together with the regularized integrals (Gel'fand and Shilov 1964)

$$\int_{-\infty}^\infty \mathrm{d}\omega\, \mathrm{e}^{\mathrm{i}\omega t}(\omega-\mathrm{i}0)^{-n} = \frac{2\pi}{\Gamma(n)}\exp(\tfrac{1}{2}\mathrm{i}n\pi)t_+^{n-1}, \qquad n = 0, 1, 2, \ldots.$$

We remark here that, to first order,

$$\langle(\Delta x_2)^2\rangle \underset{t\ll 1}{\simeq} \frac{KT}{m}t^2.$$

This is in agreement with the conventional theory of brownian motion, as one would expect.

In general only the third term on the RHS of equation (2.11) contributes to D_2. We thus have

$$D_\perp = \frac{KT\gamma^2}{2m}\int_0^\infty \mathrm{e}^{-\gamma t}|y(t)|^2\,\mathrm{d}t. \tag{2.12}$$

We have written $D_\perp$ rather than D_2 in equation (2.12) since the same expression is obtained for the coefficient of diffusion along any ray in the 1–2 plane. In fact a rotation through an angle θ about the 3 axis merely induces the transformation $\psi \to \psi+\theta$. By virtue of equation (2.9) the result follows.

In the following sections we shall use equation (2.12) to obtain $D_\perp$ for a number of specific time-dependent fields. We note here that if $\omega(t) = \omega_\mathrm{c}$ = constant, it is easy to obtain the well known result

$$D_\perp = \frac{KT}{m}\left(\frac{\gamma}{\gamma^2+\omega_\mathrm{c}^2}\right).$$

Sometimes it is useful to recast equation (2.12) into a slightly different form. Writing $g(t) = \dot{y}(t)$, it is a simple matter to obtain the following Volterra equation satisfied by $g(t)$:

$$g(t) = \frac{1}{2}+\frac{1}{2}\frac{\omega(t)}{\omega_\mathrm{c}}+\frac{\mathrm{i}}{2}\int_0^t \mathrm{d}t'(\omega(t)+\omega(t'))g(t'). \tag{2.13}$$

In terms of $g(t)$ we have

$$D_\perp = \frac{KT\gamma^2}{2m}\int_0^\infty \left|\int_0^t g(t')\,dt'\right|^2 e^{-\gamma t}\,dt$$

$$= \frac{KT\gamma^2}{4\pi m}\int_{-\infty}^\infty d\mu \left|\int_0^\infty dt\,\exp\{-(\tfrac{1}{2}\gamma+i\mu)t\}\int_0^t dt' g(t')\right|^2$$

$$= \frac{KT\gamma^2}{4\pi m}\int_{-\infty}^\infty \frac{d\mu}{\mu^2+\gamma^2/4}|\tilde{g}_+(\mu-i\gamma/2)|^2 \qquad (2.14)$$

where $\tilde{g}_+(\mu)$ is the Fourier transform of $g(t)\theta(t)$ and is thus analytic in the half plane $\mathrm{Im}(\mu) < 0$. Under certain circumstances it is possible to complete the semicircle at ∞ in the lower half plane, so that, for example, if $\tilde{g}_+^*$ has only simple poles at $\mu = \mu_n$, of residue r_n,

$$D_\perp = \frac{KT\gamma}{2m}\left(|\tilde{g}_+(-i\gamma)|^2 + \sum_n \frac{\gamma}{|\mu_n|^2}\,\mathrm{Im}\,(\tilde{g}_+(\mu_n)r_n)\right). \qquad (2.15)$$

We shall not, however, find it necessary to use equations (2.13) and (2.14) in our treatment of the following models.

3. Models without turning points

From the equations of motion (2.7) we obtain the reduced equation for the function

$$u(t) = \exp\left(-\frac{i}{2}\omega_c\int_0^t F(t')\,dt'\right)\frac{x(t)}{\dot{x}(0)}$$

$$\ddot{u}+\frac{\omega_c^2}{4}F^2(t)u = 0 \qquad (3.1)$$

$$u(0) = \frac{-i}{\omega_c}, \qquad \dot{u}(0) = \tfrac{1}{2}.$$

The solution $u(t)$ can be written

$$u = \frac{-i}{\omega_c}u_1+\frac{1}{2}u_2 \qquad (3.2)$$

where the real functions u_1, u_2 form a fundamental set for (3.1) and satisfy the boundary conditions $u_1(0) = 1, \dot{u}_1(0) = 0$ and $u_2(0) = 0, \dot{u}_2(0) = 1$. The Wronskian $W(u_1, u_2) = 1$ for all t. The zeros of u_1 and u_2 alternate and are distributed according to the magnitude of ω_c and the nature of $F(t)$. If $F(t) \to 0$ monotonically as $t \to \infty$ (or $F(t) \to \infty$ monotonically as $t \to \infty$) the solutions will be unbounded (or vanish) for large t, as is to be expected physically.

We shall mainly be interested in the behaviour of the diffusion constant as a function of ω_c, $\omega_c \gg 1$. It is thus of some interest in view of equation (2.12) to consider the asymptotic behaviour of equation (3.1) as a function of the parameter ω_c. It is known (see eg Wasow 1965) that, in general, if $\dot{F}(t) = 0$ at $t = t_0$ say, the large ω_c behaviour of $u(t)$ will differ according to whether $t \sim t_0$ or not. In this section we consider two simple models for which $F(t)$ is monotonic. We shall find that for these two cases the relation $D_\perp \propto 1/\omega_c^2$ is obeyed.

3.1. Model (i)

$$F(t) = 1+\beta t, \qquad \beta > 0. \tag{3.3}$$

Application of the criteria for the adiabatic approximation (Seymour 1966)

$$\left|\frac{\dot{B}(t)}{B^2(t)}\right| \ll 1, \qquad \left|\frac{\ddot{B}(t)}{B^3(t)}\right| \ll 1, \qquad \text{for all } t > 0$$

leads directly to the condition

$$\frac{\omega_c}{\beta} \gg 1 \tag{3.4}$$

for a slowly varying linear field. Indeed, it is easily seen that equation (3.4) is the condition required for the adiabatic approximation for all four models we consider and we shall not restate it each time.

Here,

$$\frac{x(t)}{\dot{x}(0)} = \exp\left(\frac{\mathrm{i}\omega_c t}{4}(2+\beta t)\right)u(t)$$

and the reduced equation of motion is

$$\ddot{u}+\frac{\omega_c^2}{4}(1+\beta t)^2 u = 0$$

which has the solution

$$u(t) = (1+\beta t)^{1/2}\left\{A\mathrm{J}_{1/4}\left(\frac{\omega_c}{4\beta}(1+\beta t)^2\right)+B\mathrm{N}_{1/4}\left(\frac{\omega_c}{4\beta}(1+\beta t)^2\right)\right\}$$

where $\mathrm{J}_{1/4}$, $\mathrm{N}_{1/4}$ are Bessel functions of the first and second kind respectively as defined, for example, in Gradshteyn and Ryzhik (1965). The constants A and B are given by

$$\begin{pmatrix} A \\ \mathrm{B} \end{pmatrix} = -\frac{\pi}{8\beta}\begin{pmatrix} \mathrm{N}_{1/4}-\mathrm{i}\mathrm{N}_{-3/4} \\ -\mathrm{J}_{1/4}+\mathrm{i}\mathrm{J}_{-3/4} \end{pmatrix}(\xi_0)$$

where $\xi_0 = \frac{1}{4}\omega_c/\beta$. For large ω_c/β it follows from the well known asymptotic behaviour of Bessel functions (Gradshteyn and Ryzhik 1965) that

$$u(t) \underset{\omega_c/\beta \gg 1}{\simeq} -\frac{\mathrm{i}\pi}{8\beta}\left(\frac{2}{\pi\xi_0}\right)^{1/2} \exp\left\{-\mathrm{i}\left(\xi_0-\frac{3\pi}{8}\right)\right\}(1+\beta t)^{1/2}\mathrm{H}^{(1)}_{1/4}(\xi)$$

where $\xi = \frac{1}{4}\omega_c(1+\beta t)^2/\beta$. Here $\mathrm{H}^{(1)}_{1/4}$ is the Hankel function of the first kind. Since $\xi \gg 1$ for all $t > 0$, we may use the asymptotic form of $\mathrm{H}^{(1)}_{1/4}$ to write

$$u(t) \underset{\omega_c/\beta \gg 1}{\simeq} -\frac{\mathrm{i}}{4\beta}\frac{1}{(\xi\xi_0)^{1/2}}\exp\{\mathrm{i}(\xi-\xi_0)\}(1+\beta t)^{1/2}$$

and hence

$$\frac{x(t)}{\dot{x}(0)} \underset{\omega_c/\beta \gg 1}{\simeq} -\frac{\mathrm{i}}{\omega_c}(1+\beta t)^{-1/2}\exp\left(\frac{\mathrm{i}\omega_c t}{2}(2+\beta t)\right).$$

From equation (2.12),

$$D_\perp = \frac{KT\gamma^2}{2m}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\gamma t}|y(t)|^2$$
$$= \frac{KT\gamma^2}{2m}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\gamma t}\left|\frac{x(t)}{\dot{x}(0)}+\frac{\mathrm{i}}{\omega_\mathrm{c}}\right|^2$$
$$\simeq \frac{KT\gamma^2}{2m\omega_\mathrm{c}^2}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\gamma t}\left\{1+\frac{1}{1+\beta t}-\frac{2}{(1+\beta t)^{1/2}}\cos\left(\frac{\omega_\mathrm{c}t}{2}(2+\beta t)\right)\right\}.$$

The first two terms are easily evaluated but the third requires more work. We have

$$D_\perp = \frac{KT\gamma^2}{2m\omega_\mathrm{c}^2}\left\{\frac{1}{\gamma}-\frac{1}{\beta}\exp\left(\frac{\gamma}{\beta}\right)\mathrm{Ei}\left(\frac{-\gamma}{\beta}\right)-2I\right\} \tag{3.5}$$

where Ei is the exponential integral function (see eg Gradshteyn and Ryzhik 1965) and

$$I = \int_0^\infty \mathrm{d}t\frac{\mathrm{e}^{-\gamma t}}{(1+\beta t)^{1/2}}\cos\left(\frac{\omega_\mathrm{c}t}{2}(2+\beta t)\right).$$

This may be evaluated in terms of incomplete gamma functions (Gradshteyn and Ryzhik 1965) of the form $\Gamma(\frac{1}{4}, \pm\frac{1}{2}\mathrm{i}\omega_\mathrm{c}/\beta)$ and the asymptotic behaviour of these functions for $\omega_\mathrm{c}/\beta \gg 1$ leads to the result

$$I = \mathrm{O}\left(\frac{1}{\omega_\mathrm{c}^{3/2}}\right), \qquad \omega_\mathrm{c} \gg 1.$$

Therefore, retaining only the first two terms in equation (3.5), we obtain

$$D_\perp \underset{\omega_\mathrm{c}\gg 1}{\simeq} \frac{KT\gamma^2}{2m\omega_\mathrm{c}^2}\left\{\frac{1}{\gamma}-\frac{1}{\beta}\exp\left(\frac{\gamma}{\beta}\right)\mathrm{Ei}\left(\frac{-\gamma}{\beta}\right)\right\}. \tag{3.6}$$

If $\beta \ll 1$, this is simplified further to

$$D_\perp \underset{\substack{\omega_\mathrm{c}\gg 1\\ \beta\ll 1}}{\simeq} \frac{KT\gamma}{m\omega_\mathrm{c}^2}.$$

Thus $D_\perp$ has the usual $1/\omega_\mathrm{c}^2$ dependence for large $\omega_\mathrm{c}(= B)$ in this model in which the magnetic field increases linearly with time.

3.2. *Model* (*ii*)

$$F(t) = \begin{cases} \mathrm{e}^{\beta t}, & 0 \leqslant t \leqslant \tau \\ M = \mathrm{e}^{\beta\tau}, & t \geqslant \tau. \end{cases} \tag{3.7}$$

Proceeding in the same way as for model (i) we obtain

$$\frac{x(t)}{\dot{x}(0)} = \begin{cases} \exp\{\mathrm{i}p(\mathrm{e}^{\beta t}-1)\}\{a(p)\mathrm{J}_0(p\,\mathrm{e}^{\beta t})+b(p)\mathrm{N}_0(p\,\mathrm{e}^{\beta t})\} & 0 \leqslant t \leqslant \tau \\ \exp[\mathrm{i}p\{M(t-\tau)+\beta^{-1}(M-1)\}]\{c(q)\cos(\beta qt)+d(q)\sin(\beta qt)\} & t \geqslant \tau. \end{cases}$$

Here $p = \omega_\mathrm{c}/2\beta$, $q = Mp$. The constants of integration are given by

$$\begin{pmatrix} a \\ b \end{pmatrix}(p) = \frac{\mathrm{i}\pi v_\perp \mathrm{e}^{\mathrm{i}\psi}}{4\beta}\begin{pmatrix} -\mathrm{N}_0+\mathrm{i}\mathrm{N}_1 \\ \mathrm{J}_0-\mathrm{i}\mathrm{J}_1 \end{pmatrix}(p),$$

and

$$\begin{pmatrix} c \\ d \end{pmatrix}(q) = \begin{pmatrix} \cos(\beta\tau q) & \sin(\beta\tau q) \\ \sin(\beta\tau q) & -\cos(\beta\tau q) \end{pmatrix} \begin{pmatrix} \mathrm{J}_0(q) & \mathrm{N}_0(q) \\ \mathrm{J}_1(q) & \mathrm{N}_1(q) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

For $\omega_c \gg 1$, we easily arrive at

$$y(t) \underset{\omega_c \gg 1}{\simeq} \begin{cases} \dfrac{\mathrm{i}}{\omega_c}\left\{1-\exp\left(-\dfrac{\beta t}{2}+\dfrac{\mathrm{i}\omega_c}{\beta}(\mathrm{e}^{\beta t}-1)\right)\right\} & 0 \leqslant t \leqslant \tau \\ \dfrac{\mathrm{i}}{\omega_c}[1-M^{-1/2}\exp \mathrm{i}\omega_c\{M(t-\tau)+\beta^{-1}(M-1)\}] & t \geqslant \tau. \end{cases}$$

Write

$$D_\perp = \frac{KT\gamma^2}{2m}(I_1+I_2)$$

with

$$I_1 = \int_0^\tau \mathrm{e}^{-\gamma t}|y(t)|^2\,\mathrm{d}t$$

$$I_2 = \int_\tau^\infty \mathrm{e}^{-\gamma t}|y(t)|^2\,\mathrm{d}t.$$

Then, since I_1 converges uniformly in ω_c, we have for $\omega_c \gg 1$

$$I_1 \simeq \omega_c^{-2}\left[\gamma^{-1}(1-\mathrm{e}^{-\gamma\tau})+(\gamma+\beta)^{-1}(1-M^{-1}\,\mathrm{e}^{-\gamma\tau})-2\int_0^\tau \mathrm{d}t\,\exp\left\{-\left(\gamma+\frac{\beta}{2}\right)t\right\} \times \cos\left(\frac{\omega_c}{\beta}(\mathrm{e}^{\beta t}-1)\right)\right].$$

We need to estimate the integral appearing on the RHS of the above equation for large ω_c. This integral can be cast into the form

$$\int_1^M x^{-\mu-1}\cos\lambda(x-1)\,\mathrm{d}x$$

whose asymptotic behaviour in λ can be obtained from that of

$$I(\lambda,\mu) = \int_1^M x^{-\mu-1}\,\mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}x = M^{-\mu}S(M\lambda,\mu)-S(\lambda,\mu).$$

Here we have written

$$S(\lambda,\mu) = \sum_{n=0}^\infty \frac{(\mathrm{i}\lambda)^n}{n!(n-\mu)}.$$

From a well known theorem of Barnes (Ford 1916), we have

$$S(\lambda,\mu) \underset{\lambda \gg 1}{\simeq} \Gamma(-\mu)\exp(-\tfrac{1}{2}\mathrm{i}\mu\pi)\lambda^\mu - \mathrm{i}\lambda^{-1}\,\mathrm{e}^{\mathrm{i}\lambda}+\ldots.$$

It is now straightforward substitution to obtain

$$I_1 \underset{\omega_c \gg 1}{\simeq} \omega_c^{-2}\left\{\gamma^{-1}(1-\mathrm{e}^{-\gamma\tau})+(\gamma+\beta)^{-1}(1-M^{-1}\,\mathrm{e}^{-\gamma\tau})-\frac{2\,\mathrm{e}^{-\gamma\tau}}{\omega_c M^{3/2}}\sin\left(\frac{\omega_c}{\beta}(M-1)\right)\right\}. \qquad (3.8)$$

The corresponding expression for I_2 can be obtained in an elementary way:

$$I_2 \underset{\omega_c \gg 1}{\simeq} \frac{e^{-\gamma\tau}}{\omega_c^2}\left[\gamma^{-1}(1+M^{-1})+\frac{2}{M^{1/2}(\gamma^2+\omega_c^2 M)}\left\{\omega_c M \sin\left(\frac{\omega_c}{\beta}(M-1)\right)\right.\right.$$
$$\left.\left.-\gamma\cos\left(\frac{\omega_c}{\beta}(M-1)\right)\right\}\right]. \tag{3.9}$$

Retaining only the leading term it follows from equations (3.8) and (3.9) that

$$D_\perp \simeq \frac{KT\gamma}{2m\omega_c^2}\left(\frac{(2\gamma+\beta)M+\beta\, e^{-\gamma\tau}}{(\gamma+\beta)M}\right). \tag{3.10}$$

4. Models with turning points

In this section we consider two models for which $\dot{F}(t) = 0$ for at least one $t = t_0 > 0$. The second model, where the magnetic field oscillates according to $F(t) = (1-\beta\cos 2t)^{1/2}$, $0 < \beta < 1$, is very interesting physically, but unfortunately a purely analytic calculation of $D_\perp$ is an extremely difficult problem due to the nature of the function $y(t)$. We do not carry out the necessary numerical analysis here, but content ourselves with an investigation of the asymptotic development of $y(t)$ in ω_c.

4.1. Model (i′)

$$F(t) = \frac{1}{\sqrt{2}}\{1+(\beta t-1)^2\}^{1/2}, \qquad \beta > 0. \tag{4.1}$$

The function

$$u(t) = \frac{x(t)}{\dot{x}(0)}\exp\left(\frac{i\omega_c}{2\sqrt{2}}\int_0^t \{1+(\beta t'-1)^2\}^{1/2}\, dt'\right)$$

is given by

$$u(t) = AD_{-(1+i\lambda)/2}\{(1+i)\lambda^{1/2}(\beta t-1)\}+BD_{-(1+i\lambda)/2}\{-(1+i)\lambda^{1/2}(\beta t-1)\}$$

where $\lambda = \omega_c/2\sqrt{2}\beta \gg 1$, A and B are constants and $D_{-(1+i\lambda)/2}$ is the parabolic cylinder function, defined as in Gradshteyn and Ryzhik (1965). The behaviour of $u(t)$ for large λ may be found by looking at $D_\nu(z)$ when both $|\nu|$ and $|z|$ are large (see eg Bateman (1953) for the asymptotic behaviour of confluent hypergeometric functions when the variable and parameters are large). In practice, the asymptotic dependence on λ may be seen most easily by investigation of the reduced equation of motion

$$\frac{d^2u}{d\xi^2}+\lambda^2(\xi^2+1)u = 0 \tag{4.2}$$

where $\xi = \beta t-1$, with boundary conditions

$$u(\xi = -1) = \frac{-i}{2\sqrt{2\beta\lambda}}$$

$$\frac{du}{d\xi}(\xi = -1) = \frac{1}{2\beta}.$$

Write

$$u(\xi) = (\xi^2+1)^{-1/4} \exp\left(i\lambda \int^{\xi} (\xi'^2+1)^{1/2}\, d\xi'\right) v(\xi); \tag{4.3}$$

the function $v(\xi)$ then satisfies the equation

$$\frac{d^2 v}{d\xi^2} - \frac{\xi}{\xi^2+1}\frac{dv}{d\xi} + 2i\lambda(\xi^2+1)^{1/2}\frac{dv}{d\xi} + \frac{3\xi^2-2}{4(\xi^2+1)^2} v = 0.$$

Expansion of $v(\xi)$ as a power series in $1/\lambda$ leads to the solution

$$v(\xi) = K_0 + \frac{1}{\lambda}\left(\frac{iK_0}{8}\int^{\xi}\frac{3\xi'^2-2}{(\xi'^2+1)^{5/2}}\, d\xi' + K_1\right) + O\left(\frac{1}{\lambda^2}\right) \tag{4.4}$$

where K_0 and K_1 are constants. The expression (4.4) for $v(\xi)$ gives one solution of equation (4.2) for $u(\xi)$; another independent solution is obtained by replacing λ by $-\lambda$ in equation (4.3) and following the same procedure to solve the new equation for $v(\xi)$. The complete solution for $u(\xi)$ is found to be

$$u(\xi) = -\frac{i}{\omega_c}\left(\frac{\xi^2+1}{2}\right)^{-1/4} \exp\left[\frac{i\omega_c}{4\sqrt{2}\beta}\left\{\ln\left(\frac{\xi + (\xi^2+1)^{1/2}}{\sqrt{2}-1}\right) + \xi(\xi^2+1)^{1/2} + 2\right\}\right] + O\left(\frac{1}{\omega_c^2}\right). \tag{4.5}$$

By equation (2.12) and the triangle inequality,

$$\frac{kT\gamma^2}{2m}\int_0^{\infty} dt\, e^{-\gamma t}\left||u(t)|^2 - \frac{1}{\omega_c^2}\right| \leqslant D_\perp \leqslant \frac{kT\gamma^2}{2m}\int_0^{\infty} dt\, e^{-\gamma t}\left(|u(t)|^2 + \frac{1}{\omega_c^2}\right). \tag{4.6}$$

Calculation of the diffusion coefficient thus necessitates the evaluation of

$$\int_0^{\infty} dt\, e^{-\gamma t}|u(t)|^2 = \frac{\sqrt{2}}{\omega_c^2\beta}\exp\left(\frac{-\gamma}{\beta}\right)(I_1 + I_2) + O\left(\frac{1}{\omega_c^3}\right) \tag{4.7}$$

where

$$I_1 = \int_0^{\infty}\frac{d\xi}{(\xi^2+1)^{1/2}}\exp\left(-\frac{\gamma\xi}{\beta}\right)$$

$$I_2 = \int_0^{1}\frac{d\xi}{(\xi^2+1)^{1/2}}\exp\left(\frac{\gamma\xi}{\beta}\right).$$

Both of these integrals are positive and finite; I_1 may be integrated immediately to give

$$I_1 = \frac{\pi}{2}\left\{\mathscr{H}_0\left(\frac{\gamma}{\beta}\right) - N_0\left(\frac{\gamma}{\beta}\right)\right\} \tag{4.8}$$

where $\mathscr{H}_0$ is a Struve function and N_0 is a Bessel function of the second kind (see Gradshteyn and Ryzhik 1965). I_2 cannot be expressed easily in terms of special functions. However, it clearly lies within the limits

$$\frac{\beta}{\sqrt{2}\gamma}\left\{\exp\left(\frac{\gamma}{\beta}\right) - 1\right\} \leqslant I_2 \leqslant \frac{\beta}{\gamma}\left\{\exp\left(\frac{\gamma}{\beta}\right) - 1\right\}. \tag{4.9}$$

Finally, from equations (4.6), (4.7), (4.8) and (4.9), it follows that

$$D_\perp \underset{\omega_c \gg 1}{\simeq} \frac{\text{constant}}{\omega_c^2} \tag{4.10}$$

which is the classical $1/B^2$ dependence once more.

4.2. Model (ii′)

$$F(t) = (1-\beta \cos 2t)^{1/2}, \qquad 0 < \beta < 1. \tag{4.11}$$

The reduced equation of motion satisfied by

$$u(t) = \frac{x(t)}{\dot{x}(0)} \exp\left(\frac{-i\omega_c}{2}\int_0^t (1-\beta\cos 2t')^{1/2}\, dt'\right)$$

is

$$\ddot{u} + \frac{\omega_c^2}{4}(1-\beta \cos 2t)u = 0 \tag{4.12}$$

which is the standard Mathieu equation

$$\ddot{u} + (\Delta - \Omega \cos 2t)u = 0 \tag{4.13}$$

with $\Delta = \omega_c^2/4, \Omega/\Delta = \beta$. For the purpose of the present discussion, we use the notation of Langer (1934). The solutions of equations (4.13) belong to three distinct classes, namely those of periodic, bounded and unbounded functions, corresponding to the regions of the $\Delta - \Omega$ plane sketched in figure 1. In particular, the solutions of equation (4.12) lie on the line $\Delta = \Omega/\beta$, $\beta < 1$. The interesting physical ramifications of the parametric resonance effect, obtained by varying ω_c, are well known and have been discussed elsewhere in considerable detail (McLachlan 1947, see also Landau and Lifshitz 1960).

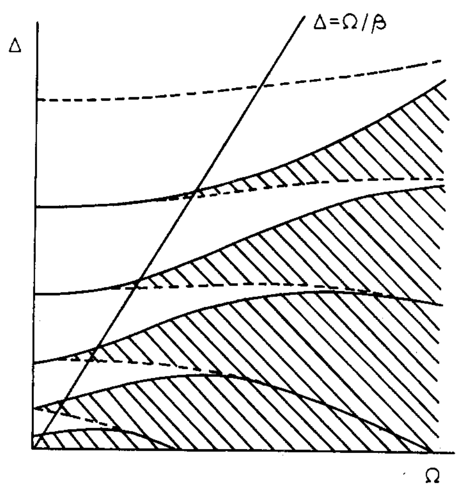

Figure 1. Stability regions in the $\Delta - \Omega$ plane. The shaded areas correspond to unstable solutions, the open areas to stable solutions. These regions are separated by boundaries which correspond to periodic solutions; full lines correspond to even periodic solutions, broken lines to odd periodic solutions. The solutions of equation (4.12) lie on the line $\Delta = \Omega/\beta$.

A fundamental system for equation (4.12) is provided by the functions $u_0(t)$, $u_e(t)$ satisfying $u_0(0) = 0$, $\dot{u}_0(0) = 1$, $u_e(0) = 1$, $\dot{u}_e(0) = 0$. In terms of these functions, we have

$$u(t) = \tfrac{1}{2}u_0(t) - \frac{i}{\omega_c}u_e(t). \tag{4.14}$$

From the work of Langer (1934) we can deduce the following results: for $0 \leqslant t \leqslant \pi/2$,

$$u_0(t) \underset{\omega_c \gg 1}{\simeq} \frac{2 \sin \alpha(t)}{\omega_c\{(1-\beta)(1-\beta \cos 2t)\}^{1/4}}$$

$$u_e(t) \underset{\omega_c \gg 1}{\simeq} \left(\frac{1-\beta}{1-\beta \cos 2t}\right)^{1/4} \cos \alpha(t)$$

where

$$\alpha(t) = \frac{\omega_c}{2} \int_0^t (1-\beta \cos 2t')^{1/2} \, dt'.$$

Hence the solution of equation (4.12) is

$$u(t) \underset{\omega_c \gg 1}{\simeq} \frac{-i\{(1-\beta)^{1/2} \cos \alpha(t) + i \sin \alpha(t)\}}{\omega_c\{(1-\beta)(1-\beta \cos 2t)\}^{1/4}} + O\left(\frac{1}{\omega_c^2}\right), \qquad 0 \leqslant t \leqslant \pi/2. \tag{4.15}$$

Alternative representations are required for $\pi/2 < t \leqslant 3\pi/2$, $3\pi/2 < t \leqslant 5\pi/2, \ldots$ and these can be built up from equation (4.15) by making use of the complete set of functional relations given by Langer (1934).

The integral (2.12) for $D_\perp$ needs to be computed numerically; this computation has not yet been performed. We expect that the behaviour (4.15) of $u(t)$ will manifest itself in the B dependence of the diffusion coefficient in the strong field limit. This interesting problem will be developed further elsewhere.

5. Conclusion

In the adiabatic approximation, the classical $1/B^2$ dependence of the perpendicular diffusion coefficient in the strong field limit has been obtained for models in which the magnetic field is a monotonic function of time and for models in which the field possesses a turning point. We have seen that the asymptotic dependence on $\omega_c(= B)$ is not altered in the neighbourhood of a turning point. The $1/B^2$ dependence agrees with the results of many experiments with plasmas in which fluctuations of the density are small (see eg D'Angelo and Rynn 1961, Simon 1958), but offers no explanation of experiments such as those reported by Hoh and Lehnert (1960), Bonnal *et al* (1961) and Chen and Bingham (1961), where there is anomalous diffusion inversely proportional to B.

Preliminary calculations indicate that the asymptotic behaviour in ω_c may be affected by the presence of a zero in the magnetic field itself, rather than in its derivative. For example, in the case of a sinusoidally varying field $B(t) = \omega_c \cos \beta t$, the Mathieu function solutions of the reduced equation of motion behave like $1/\omega_c^{3/4}$ in the neighbourhood of a zero of $B(t)$ and this could possibly give rise to $1/\omega_c^{3/2}$ dependence of the diffusion coefficient. The suggestion (see eg Spitzer 1960) that fluctuating fields may provide an explanation on the microscopic level for anomalous diffusion is not inconsistent with such a possibility. However, when $B(t)$ has zeros the Seymour conditions for the adiabatic approximation no longer hold and a new formalism must be developed in order to make a rigorous study of the diffusion coefficient.

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